A Hybrid Method for Solving Tridiagonal Systems on GPU

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The Hybrid Method
A comparison between the CR, PCR and hybrid algorithms in terms of algorithmic steps and work per step for solving an 8-equation system. A dot stands for a unit of work (an equation), and a row of dots stands for an algorithmic step. We assume the length of vector arithmetic unit is 4 in this example.

The Thomas Algorithm
2n = 16 steps. The vector unit has a utilization rate of 25% through all steps. Less work per step, more steps.

Parallel Cyclic Reduction (PCR)
log2(8) = 3 steps. The vector unit is fully utilized across all steps. More work per step, fewer steps.

Cyclic Reduction (CR)
2log2(8) = 6 steps. The vector unit is partially idle during the middle three steps. Less work per step, more steps.

CR-PCR
2 + 2log2(4) = 4 steps. The vector unit is fully utilized across all steps. Less work per step than PCR, fewer steps than CR.

PCR-Thomas
2 + 2n = 6 steps. The vector unit is fully utilized across all steps. Less work per step than PCR, fewer steps than the Thomas algorithm.

Problem Statement

\[
\begin{bmatrix}
 b_1 & c_1 & 0 & \cdots & 0 \\
 a_2 & b_2 & c_2 & \cdots & 0 \\
 0 & a_3 & b_3 & \cdots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & \cdots & 0 & a_{n-1} & b_{n-1} \\
 0 & \cdots & 0 & 0 & a_n \\
\end{bmatrix}
\begin{bmatrix}
 x_1 \\
 x_2 \\
 \vdots \\
 x_{n-1} \\
 x_n \\
\end{bmatrix}
= \begin{bmatrix}
 d_1 \\
 d_2 \\
 \vdots \\
 d_{n-1} \\
 d_n \\
\end{bmatrix}
\]

Algorithm | Algorithmic steps | Work per step
--- | --- | ---
The Thomas Algorithm | 2n | 1
Cyclic Reduction | 2 log2 n | \( \frac{n}{2} \)
Parallel Cyclic Reduction | log2 n | n

Numerous Applications
- Shallow Water Simulation
- Depth-of-fields Blurs
- Numerical Ocean Models
- Spectral Poisson Solvers
- Cubic Spline Approximation
- Semi-coarsening for Multi-grid Solvers
- Alternating Direction Implicit (ADI) Method
- Pre-conditioners for Iterative Linear Solvers

Reference